PMM U.S.S.R.,Vol.45,pp.839-847 0021-8928/82/6 0839 $7,50/0

UDC 624.07:534.1

ON THE STABILITY OF INHOMOGENEOUSLY VISCOELASTIC REINFORCED BARS*

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There is investigated the stability of inhomogeneously ageing reinforced viscoelastic bars. It is assumed that the strains and stresses in the reinforcement are related by Hooke's law. The properties of the matrix material are described by equations of the theory of viscoelasticity of inhomogeneously ageing solids /1,2/. Under different boundary conditions for the ends of the bar and loading methods an expression is set up for the critical force in stability problems in an infinite time interval. The stability definition taken corresponds to the Liapunov stability definition for the motion of dynamical systems. Estimates of the critical time when the magnitude of the deflection of a viscoelastic bar reaches a given value are obtained for stability problems in a finite time interval. The formulation for the stability problem in a finite time interval starts from the definition of stability of motion of dynamical systems by taking its beginning from the Chetaev work. The dependence of the critical time on the inhomogeneity and the reinforcing parameter is investigated numerically. The stability of viscoelastic unreinforced bars was studied in /3,4/, A survey and bibliography of research associated with the stability problem for viscoelastic bars are available in /5-8/.

1. Model of an inhomogeneously ageing viscoelastic solid. Its specific inhomogeneity characterizes the model of an inhomogeneously ageing viscoelastic solid with time-varying elastic and rheological properties. This inhomogeneity is due to the fact that the natural and artificial ageing processes in such a solid proceed dissimilarly in all its elements. Hence, the age of the material generally depends on the spatial coordinates, which, in turn, determines the form of the function characterizing the properties of the viscoelastic solid as a function of the time and spatial coordinates.

The model mentioned can describe the processes of discrete and continuous accumulation of elements of different age by viscoelastic solids. The equation of state relating the strain $\varepsilon_x(t)$ to the stress $\sigma_x(t)$ for an inhomogeneously ageing solid in the uniaxial state of stress has the following form:

$$\varepsilon_x(t)=\frac{\sigma_x(t)}{E(t+\rho(x))}-\int_{t_0}^{t}\sigma_x(s)K(t+\rho(x),\ s+\rho(x))\,ds \tag{1.1}$$

$$K(t,\tau)=\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)}+N(t,\tau)\right] \tag{1.2}$$

Here $K(t,\tau)$ is the creep kernel for a homogeneously ageing solid, $E(t)$ is the variable modulus of instantaneous elastic strain, $N(t,\tau)$ is the measure of the material creep, t_0 is the time of stress application, $\rho(x)$ is the age of an element with coordinate x relative to an element with the coordinate $x=0$.

2. Equation for the deflection of a reinforced viscoelastic bar. Let us derive the equation for the deflections of a reinforced inhomogeneously viscoelastic bar under the following hypotheses:

1) The strains and stresses in the reinforcement satisfy Hooke's law, and the relationships (1.1) and (1.2) in the main material;

2) The center of gravity of the reinforcement in each section of the bar agrees with the center of gravity of the main material; the bar cross sections are identical and are oriented identically;

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3) The transverse sections of the bar elements remain planar during deformation and the law of plane sections is valid.

Let us insert a Cartesian coordinate system with origin O at the center of gravity of the transverse section in the bar. We take the line of intersection of the bar bending plane with the plane of the transverse section as the z axis. The z_1 axis agrees with the neutral axis (see Fig.1). The trace of the reinforcement is shown by the solid lines in Fig.1.

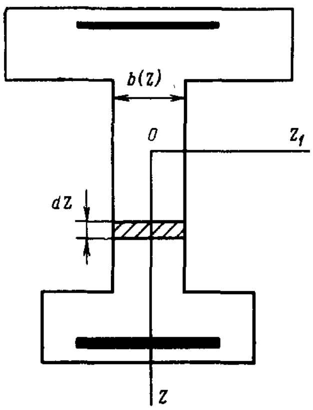

Fig.1

By virtue of the law of plane sections we have

$$\varepsilon_x(t, z) = \omega(t)z \tag{2.1}$$

where $\omega(t)$ is the curvature of the bar neutral line at the time t.

We take the viscoelasticity law for the main material in the form of (1.1) and (1.2) for a constant modulus of instantaneous elastic strain E and the creep measure $N(t, \tau)$ satisfying the equality

$$N(t, \tau) = \varphi(\tau)[1 - e^{-\gamma(t-\tau)}] \tag{2.2}$$

In conformity with Hooke's law, we will have for the i-th element of the reinforcement

$$\sigma_a^i(t) = E_a \varepsilon_a^i(t) \tag{2.3}$$

The equilibrium equation yields

$$\sum_{i=1}^{n} F_a^i \sigma_a^i z_i + \int_F \sigma_x(t, z) z b(z)\, dz = M(t, x) \tag{2.4}$$

Here F_a^i is the cross-sectional area of the i-th element in the reinforcement, F is the area of the main part of the section, and $M(t,x)$ is the bending moment in the section x.

It follows from the relationships (2.1) and (1.1) that

$$\sigma_x(t, z) = Ez\left[\omega(t) + \int_{t_0}^{t} \omega(\tau) R(t + \rho(x), \tau + \rho(x))\, d\tau\right] \tag{2.5}$$

Here $R(t, \tau)$ is the resolvent of the kernel $EK(t, \tau)$. Substituting (2.3) and (2.5) into the equilibrium equation (2.4), we obtain

$$\omega(t) + \beta \int_{t_0}^{t} \omega(\tau) R(t + \rho(x), \tau + \rho(x))\, d\tau = -\frac{1}{JE} M(t, x) \tag{2.6}$$

$$J = E^{-1}(E_a J_a + EJ_1), \quad \beta = J_1 J^{-1}$$

Here J_a is the moment of inertia of the reinforcement relative to the axis Oz_1 and J_1 denotes the moment of inertia of the transverse section of the main material relative to the axis Oz_1.

Furthermore, because of (/9/, p.185) and (1.2) and (2.2), we have

$$R(t, \tau) = -\gamma\varphi(\tau)E + [\gamma^2\varphi(\tau)E + \gamma^2\varphi^2(\tau)E^2 + \gamma\varphi'(\tau)E]\int_{\tau}^{t} e^{-\eta(s)+\eta(\tau)}\, ds, \quad \eta(t) = \gamma\int_{t_0}^{t}[1 + E\varphi(s)]\, ds \tag{2.7}$$

Moreover

$$\omega(t) = y''(t, x) - y_0''(x), \quad y'(t, x) = \partial y(t, x)/\partial x \tag{2.8}$$

Here $y(t, x)$ is the deflection of the neutral axis, the coordinate is x, $0 \leqslant x \leqslant l$, and $y_0(x)$ is the initial deflection of the bar. We substitute (2.7) and (2.8) into (2.6) and differentiate what is obtained twice with respect to t. We obtain an equation for the deflection

$$0 = y^{\cdot\cdot\prime\prime}(t, x) + \gamma[1 + E\varphi(t + \rho(x))(1 - \beta)]\, y^{\cdot\prime\prime}(t, x) + \frac{1}{EJ} M^{\cdot\cdot}(t, x) + \frac{\gamma}{EJ} M^{\cdot}(t, x)[1 + E\varphi(t + \rho(x))] \tag{2.9}$$

$$y^{\cdot}(t, x) = \partial y(t, x)/\partial t, \quad t \geqslant t_0$$

For unreinforced bars (i.e., for $\beta = 1$), the equation (2.9) has been presented earlier in /4/. The magnitude of the deflection $y(t_0, x)$ at the initial time t_0 directly after application of the stress will satisfy the equation

$$y''(t_0, x) + \frac{1}{EJ} M(t_0, x) = y_0''(x) \tag{2.10}$$

The rate of change of the deflection at the initial time (i.e., the derivative $y^{\cdot}(t, x)$ at $t = t_0$) satisfies the equation

$$y^{\cdot\prime\prime}(t_0, x) + \frac{1}{EJ} M^{\cdot}(t_0, x) = -\frac{\gamma\beta}{J}\varphi(t_0 + \rho(x)) M(t_0, x) \tag{2.11}$$

In order to determine the quantity $y(t_0, x)$ and $y^{\cdot}(t_0, x)$ from (2.10) and (2.11), specific methods for loading the bar and conditions for fixing its ends must be given to determine the bending moment $M(t, x)$ and the boundary conditions.

3. Stability in an infinite time interval. Since methods to investigate the stability are quite similar for different situations, the stability of a bar whose lower end ($x = l$) is framed while the upper is free subjected to a distributed load, will be studied in detail below. With respect to the remaining cases, we shall limit ourselves to the formulation of the problem and the stability conditions.

The bar is placed along the Ox axis in the undeformed state. The deflection of the bar $y(t, x)$ at the point x at the time $t \geqslant t_0$ is measured from the Ox axis.

The bar is called stable if for any $\delta_1 > 0$ there is a $\delta_2 > 0$ such that for $t \geqslant t_0$ and $0 \leqslant x \leqslant l$ there will be

$$\sup_{t,x} |y(t, x)| < \delta_1 \quad \text{when} \quad \sup_x |y_0(x)| < \delta_2 \tag{3.1}$$

The equation for the deflection has the form (2.9). The inhomogeneous ageing function $\rho(x)$ in this equation is assumed bounded, piecewise-continuously differentiable, and with a finite number of points of discontinuity for the derivative.

From the boundary conditions of the bar ends we have the following equations:

$$y(t, 0) = 0, \quad y''(t, 0) = 0, \quad y'(t, l) = 0 \tag{3.2}$$

The function φ is positive, continuously differentiable, and

$$\lim \varphi(t) = C_0, \quad \lim \varphi^{\cdot}(t) = 0, \quad t \to \infty, \quad C_0 > 0 \tag{3.3}$$

The bar is subjected to a constant longitudinal load of magnitude g. The bending moment $M(t, x)$ in the section x is given by the expression

$$M(t, x) = g\int_0^x (y(t, x) - y(t, z))\, dz \tag{3.4}$$

Let us introduce the sequence $\psi_n(x)$ of eigenfunctions and the sequence λ_n of eigenvalues of the boundary value problem

$$\psi_n''(x) + \lambda_n x \psi_n(x) = 0, \quad \psi_n(l) = 0, \quad \psi_n'(0) = 0 \tag{3.5}$$

It is known that the functions $\psi_n(x)$ are orthonormal in the generalized sense, i.e.,(see /10/, for instance)

$$\int_0^l \psi_n(x)\psi_m(x)\, x\, dx = 0, \quad \delta_{nm} = 0, \quad n \neq m, \quad \delta_{nm} = 1, \quad n = m \tag{3.6}$$

Let the derivative $y'(t, x)$ be represented as an absolutely and uniformly convergent series in the functions $\psi_n(x)$:

$$y'(t, x) = \sum_{n=0}^{\infty} A_n(t)\psi_n(x), \tag{3.7}$$

$$A_n(t) = \int_0^l \psi_n(x) y'(t, x)\, x\, dx \tag{3.8}$$

We substitute (3.4) and (3.7) into (2.9), we differentiate both sides of the equality obtained with respect to x, and then multiply by $\psi_n(x)$ and integrate with respect to x between zero and l. We consequently obtain

$$\mu_n(A^{\cdot\cdot}{}_n + \gamma A_n^{\cdot}) + \gamma E C_0 A_n^{\cdot} + \gamma E \sum_{m=0}^{\infty} A_m^{\cdot}\beta_{mn}(t) + \alpha_{mn} = 0, \tag{3.9}$$
$$\mu_n = 1 - EJg^{-1}\lambda_n$$

$$\bar{\beta}_{mn} = \int_0^l x\psi_m(x)\psi_n(x)\varphi(t+\rho(x))\,dx + \int_0^l \psi_n(x)\Big[\int_0^x dz \int_z^x \psi_m(s)\,ds\Big] d_x\varphi(t+\rho(x)) = \beta_{mn} + C_0\delta_{mn}$$

$$\alpha_{mn} = \gamma E(1-\beta)\sum_{m=0}^{\infty} A_m^{\cdot}\int_0^l \psi_n(x)\,d_x[\varphi(t+\rho(x))\psi_m'(x)] \qquad (3.10)$$

Let us manipulate the expression for α_{mn}. Integrating twice by parts, and taking account of the boundary conditions (3.5), we obtain

$$\int_0^l \psi_n(x)\,d_x[\varphi(t+\rho(x))\psi_m'(x)] = -C_0\delta_{mn}\lambda_n - \lambda_n\int_0^l x\psi_n(x)\psi_m(x)(\varphi(t+\rho(x))-C_0)\,dx +$$

$$\int_0^l \psi_m(x)\psi_n'(x)\,d_x\varphi(t+\rho(x)) = -C_0\delta_{mn}\lambda_n + \alpha_{mn}^{(1)}$$

Hence, and from (3.10), there results

$$\alpha_{mn} = -\gamma E(1-\beta)\Big(C_0\lambda_n A_n^{\cdot} - \sum_{m=0}^{\infty}\alpha_{mn}^{(1)}A_m^{\cdot}\Big) \qquad (3.11)$$

We now establish that the sufficient condition for stability is

$$g < \frac{JE\lambda_0}{1+EC_0}(1+EC_0(1-\beta)), \quad \lambda_0 = 7.8373 l^{-3} \qquad (3.12)$$

Here λ_0 is the minimal eigenvalue of the boundary value problem (3.5).

This stability condition for unreinforced bars (i.e., for $\beta = 1$) is established in /4/. If the percentage of reinforcement is increased, where $\beta \to 0$ (i.e., an elastic bar is obtained in the limit), then the known stability condition for an elastic bar subjected to a distributed load $g < JE\lambda_0$ is obtained from (3.12) (see /11/, for example). There results from (3.12) that the critical length l_0 of a viscoelastic reinforced bar equals

$$l_0 = 1.9863\,[JEg^{-1}(1 - EC_0\beta(1+EC_0)^{-1})]^{1/3}$$

Because of (3.12), an increase in the critical length l_0 of a reinforced viscoelastic bar as compared with the critical length l_1 of the corresponding unreinforced bar is described by the expression $l_0^3 = l_1^3(1+(1-\beta)EC_0)$.

Let us examine another limiting case in which the main material is assumed elastic with the elastic modulus E. Let l_2 denote the critical length of the reinforced bar for which both the reinforcement and the main material are subject to Hooke's law. It is clear that the quantity l_2 should be greater than the critical length of the corresponding viscoelastic reinforced bar. On the basis of (3.12), the dependence between l_0 and l_2 has the form

$$l_0 = l_2\Big(1 - \frac{EC_0\beta}{1+EC_0}\Big)^{1/3}$$

Let us turn to giving a foundation for the stability condition (3.12). We consider the function

$$V(t) = \sum_{m=0}^{\infty} A_m^{\cdot}(t)^2$$

In conformity with (3.9)–(3.11) we have

$$V^{\cdot}(t) = 2\gamma\sum_{n=0}^{\infty} A_n^{\cdot 2}\Big[-1 - \frac{EC_0}{\mu_n} + \frac{EC_0(1-\beta)\lambda_n}{\mu_n}\Big] - 2\gamma E\sum_{n=0}^{\infty} A_n\mu_n\sum_{m=0}^{\infty}\beta_{mn}A_m^{\cdot} + Q \qquad (3.13)$$

$$Q = -2\gamma E(1-\beta)\sum_{n=0}^{\infty} A_n^{\cdot}(t)\mu_n\sum_{m=0}^{\infty}\alpha_{mn}^{(1)}A_m^{\cdot}(t)$$

We examine the separate components in the right side of (3.13). The eigenvalues λ_n of the boundary value problem (3.5) are determined from the equation $J_{-1/3}(2/3\sqrt{\lambda_n l^3}) = 0$, where $J_{-1/3}$ is the Bessel function of the first kind of order $-1/3$ (see /11/, for example). Consequently, the numbers λ_n satisfy the asymptotic equality /12/

$$\lambda_n = C_1 n^2, \quad n \to \infty, \quad C_1 > 0 \tag{3.14}$$

This means that the maximum of the expression in the square brackets in (3.13) is achieved for the minimal eigenvalue λ_0. The inequality (3.12) indeed results from the requirement of negativity of this maximum.

Furthermore, as is proved in /4/, the following formula is valid

$$\left| 2\gamma E \sum_{n=0}^{\infty} A_n^{\cdot}(t)\, \mu_n \sum_{m=0}^{\infty} A_m^{\cdot} \beta_{mn} \right| \leqslant V(t) f(t) \tag{3.15}$$

Here and henceforth, $f(t)$ denotes certain different continuous non-negative functions such that

$$\lim_{t \to \infty} f(t) = 0 \tag{3.16}$$

Finally, we turn to an estimate of the quantity Q. We let C denote certain distinct non-negative constants. We note initially that

$$\left| \sum_{n=0}^{\infty} A_n^{\cdot}(t)\, \mu_n \sum_{m=0}^{\infty} A_m^{\cdot}(t) \int_0^l \psi_m(x)\, \psi_n'(x)\, d_x \varphi(t + \rho(x)) \right| \leqslant \tag{3.17}$$

$$CV(t) \sum_{n=0}^{\infty} \mu_n^2 \int_0^l (\psi_n')^2 (\varphi')^2 \, dx \leqslant C f(t) V(t) \sum_{n=0}^{\infty} \mu_n^2 \lambda_n \leqslant f(t) V(t)$$

We now estimate the expression

$$Q_1 = \left| \sum_{n=0}^{\infty} A_n^{\cdot}(t)\, \mu_n \lambda_n \sum_{m=0}^{\infty} \int_0^l x \psi_n(x)\, \psi_m(x) \left(\varphi(t + \rho(x)) - C_0 \right) dx \right| \tag{3.18}$$

We represent the product $\mu_n \lambda_n$ in the form

$$\mu_n \lambda_n = -\frac{g}{JE} \left[1 + \left(1 - \frac{g}{JE\lambda_r} \right)^{-1} \frac{g}{JE\lambda_n} \right] \tag{3.19}$$

We substitute (3.19) into (3.18) and estimate the two components resulting here. By virtue of the Parseval equality /10/ the first is estimated as follows:

$$Q_2 = \left| \frac{g}{JE} \int_0^l y^{\cdot 2}(t, x)\, x \left(\varphi(t + \rho(x)) - C_0 \right) dx \right| \leqslant f(t) V(t)$$

Because of the asymptotic (3.14), the second component is estimated similarly to (3.17). This means $Q_1 \leqslant f(t)\, V(t)$. Therefore, we finally conclude that

$$V^{\cdot}(t) \leqslant 2\gamma \left[-1 + \frac{EC_0 - EC_0(1-\beta)\lambda_0}{EJ\lambda_0 g^{-1} - 1} + f(t) \right] V(t) \tag{3.20}$$

Now we estimate $V(t_0)$. The initial conditions for the system (3.9) are given by the following formulas that result from (2.10) and (2.11):

$$A_n(t_0) = -A_n^{\circ} \lambda_n E J g^{-1} \mu_n^{-1}, \quad A_n^{\circ} = \int_0^l y_0'(x)\, \psi_n(x)\, x \, dx \tag{3.21}$$

$$A_n^{\cdot}(t_0) = -E\gamma \mu_n^{-1} \sum_{m=0}^{\infty} A_m(t_0) \left(\beta_{mn}(t_0) + C_0 \delta_{mn} \right)$$

From the Parseval equality and (3.21) there results that

$$\sum_{n=0}^{\infty} A_n^2(t_0) \leqslant C_1 \int_0^l x \left(y_0'(x) \right)^2 dx \tag{3.22}$$

Furthermore, taking account of (3.10) and the boundary conditions (3.5) we have

$$\sum_{m=0}^{\infty} \bar{\beta}_{mn}^2(t) = \int_0^l x\, dx \left(\int_x^l \varphi(t + \rho(x_1)\, d\psi_n(x_1) \right)^2 \leqslant \varphi_1(t)\, \lambda_n, \quad \varphi_1(t) = \frac{l^3}{2} \max_{0 \leqslant x \leqslant l} \varphi(t + \rho(x))^2 \tag{3.23}$$

Formulas (3.21)—(3.23) yield

$$V(t_0) \leqslant E^2\gamma^2\varphi_1(t_0) C_1 \sum_{n=0}^{\infty} \mu_n^{-2}\lambda_n \int_0^l x y_0'(x)^2\, dx \tag{3.24}$$

There results from (3.24) that $V(t_0) < \infty$. Hence, and from (3.16) it follows that

$$V(t) \leqslant C \exp(-Ct), \quad t \geqslant t_0, \quad C > 0 \tag{3.25}$$

Let $z(t, s)$ denote the function

$$z(t, s) = \exp\left[-\gamma(t-s) - E(1-\beta)\int_s^t \varphi(\tau + \rho(x))\, d\tau\right] \tag{3.26}$$

By virtue of (2.9) we have

$$y^{\cdot\prime\prime}(t, x) = z(t, t_0)\left[y^{\cdot\prime\prime}(t_0, x) + \frac{1}{EJ} M^{\cdot}(t_0, x)\right] - \frac{1}{EJ} M^{\cdot}(t, x) - \frac{\gamma\beta}{J}\int_{t_0}^{t} M^{\cdot}(s, x)\,\varphi(s + \rho(x))\, z(t, s)\, ds \tag{3.27}$$

Because of (3.26) the first component in the right side of (3.27) decreases exponentially as $t \to \infty$. An analogous conclusion is valid with respect to the remaining components in (3.27). To prove this we note that by virtue of (3.4) and the boundary conditions (3.2) and the Parseval equality we will have

$$|M^{\cdot}(t, x)| = g\left|\int_0^x s y^{\cdot\prime}(t, s)\, ds\right| \leqslant g\frac{l^2}{2}\left[\int_0^l s\,|y^{\cdot\prime}(t, s)|^2\, ds\right]^{1/2} = \frac{gl^2}{2} V(t)^{1/2}$$

Hence, and from (3.27) and (3.25) we finally conclude that

$$|y^{\cdot\prime\prime}(t, x)| \leqslant C_1 \exp(-Ct), \quad C > 0, \quad t \geqslant t_0 \tag{3.28}$$

But

$$y(t, x) = y(t_0, x) + \int_0^t\int_0^l G(x, \xi)\, y^{\cdot\prime\prime}(s, \xi)\, ds\, d\xi \tag{3.29}$$

$$G(x, \xi) = x, \quad x \leqslant \xi; \quad G(x, \xi) = \xi, \quad x \geqslant \xi$$

This means that because of (3.28) and (3.24) the estimate (3.1) will hold. The sufficiency of the conditions (3.12) is thereby established for the stability of a bar in an infinite time interval.

It can analogously be proved /4/ that a viscoelastic bar is unstable if the condition (3.12) is violated.

4. Stability in a finite time interval. Different formulations are possible for the stability of a bar in a finite time interval. Let us examine two of them by considering an elastic bar stable, i.e., by considering that $g \leqslant JE\lambda_0$.

1°. There is a finite time interval $[0, T]$, where T is a given number. It is required to determine the conditions that must be satisfied in the time interval $[0, T]$ for the deflection $y(t, x)$ not to exceed a given critical value y^* for any x, i.e.,

$$\sup_t \sup_x |y(t, x)| \leqslant y^*, \quad t \in [0, T], \quad 0 \leqslant x \leqslant l \tag{4.1}$$

To obtain the stability conditions it is sufficient to estimate the left side of the relationship (4.1). One of the estimation methods is based on the representation (3.29) and formulas (3.25)—(3.28) in which the quantities therein are to be expressed in terms of the initial characteristics of the problem. Such a method of operation is realized in detail for unreinforced bars /3,4/. Here we present another method of estimating the deflection which is valid for more general kernels K in the equation of state (1.2). It follows from (2.6) that

$$\omega(t) = -\frac{1}{JE} M(t, x) + I, \quad I = \int_{t_0}^{t} M(\tau, x)\, K_1(\beta, t + \rho(x), \tau + \rho(x))\, d\tau \tag{4.2}$$

where K_1 is the resolvent of the kernel $-\beta R(t + \rho(x), \tau + \rho(x))$. We differentiate both sides of

(4.2) with respect to x. We obtain

$$y'(t,x)=\int_0^l G(x,\xi)\left[y_0'''(\xi)+\frac{d}{d\xi}I(t,\xi)\right]d\xi \tag{4.3}$$

Here $G(x, \xi)$ is the Green's function of the problem (3.5) for $\lambda_n = g(JE)^{-1}$. We furthermore have

$$\left|I(t,l)-\int_0^l I(t,\xi)\frac{\partial G(x,\xi)}{\partial\xi}d\xi G(x,l)\right|\leqslant g\left|\int_{t_0}^t d\tau\int_0^l\int_0^\xi(y(\tau,\xi)-y(\tau,\xi_1))\,d\xi_1\frac{\partial G(x,\xi)}{\partial\xi}\times\right. \tag{4.4}$$

$$\left.K_1(\beta,t+\rho(\xi),\tau+\rho(\xi))\,d\xi\right|\leqslant\left|\int_{t_0}^t y_1(\tau)K_3(\beta,x,\tau)\,d\tau\right|$$

Taking account of (4.3) we have

$$y_1(t)\leqslant\int_{t_0}^t y_1(\tau)K_4(\beta,\tau)\,d\tau+q_0,\quad q_0=\int_0^l\int_0^l|G(x,\xi)\,y_0'''(\xi)|\,d\xi\,dx \tag{4.5}$$

$$y_1(\tau)=\max_x|y(\tau,x)|,\quad K_3(\beta,x,\tau)=\max_{0\leqslant t\leqslant T}K_2(\beta,t,x,\tau)$$

$$K_2(\beta,t,x,\tau)=2l\int_0^l|\xi\partial G(x,\xi)/\partial\xi|\,|K_1(\beta,t+\rho(\xi),\tau+\rho(\xi))|\,d\xi$$

$$K_4(\beta,\tau)=\int_0^l K_3(\beta,x,\tau)\,dx$$

From (4.5) and the Gronwohl— Bellman lemma, we have the estimate

$$y_1(t)\leqslant q_0\exp\int_0^t K_4(\beta,\tau)\,d\tau$$

Comparing this with (4.1), we conclude that a sufficient condition for stability in the sense of validity of the inequality (4.1) will have the form

$$q_0\exp\int_{t_0}^T K_4(\beta,\tau)\,d\tau\leqslant y^* \tag{4.6}$$

2°. Another formulation of the stability problem in a finite time interval is the following. The magnitude y^* of the allowable limit value of the deflection is known. Determine the critical time t_1 at which the maximum value of the deflection first equals y^*, i.e., $t\geqslant t_0$, $0\leqslant x\leqslant l$:

$$\max_t\bar{y}(t)=y^*,\quad \bar{y}(t)=\max_x|y(t,x)| \tag{4.7}$$

Let us estimate the value of the critical time t_1. It can be shown similarly to (4.5) that

$$y_1(t)\leqslant q_0+\int_{t_0}^t y_1(\tau)\int_0^l K_2(\beta,t,x,\tau)\,dx \tag{4.8}$$

Let R_2 denote the resolvent of the kernel

$$\int_0^l K_2(\beta,t,x,\tau)\,dx$$

It follows from (4.8) that

$$y_1(t)\leqslant f_1(t),\quad f_1(t)=q_0+q_0\int_0^t R_2(\beta,t,\tau)\,d\tau$$

This means that on the basis of (4.8) the critical time $t_1\geqslant t_1^-$, where t_1^- is the least root of the equation $f_1(t)=y^*$, $t\geqslant t_0$.

The solution of equation (2.6) under the conditions (3.2) and (3.4) is necessary for a numerical investigation of the formulated stability problem.

Let the bar consist of two pieces of identical length and let the age be constant within each piece.

The following numerical values of the parameters are general in the computations:

$$\varphi(\tau) = C + A\exp(-\beta_1\tau),\quad l = 1\,\mathrm{m},\quad y_0'' = -2\cdot10^{-3}\,\mathrm{m}^{-1},\quad \gamma = 0.006\,\mathrm{day}^{-1},\quad \beta_1 = 0.031\,\mathrm{day}^{-1},$$

$$C = 0.2761\cdot10^4\,\mathrm{MPa}^{-1},\quad A = 10^{-4}\,\mathrm{MPa}^{-1},\quad E = 3.3\cdot10^4\,\mathrm{MPa},\quad y^* = 2.2\cdot10^{-3}\,\mathrm{m},\quad g/JE = 3\cdot10^{-2}\,\mathrm{m}^{-}$$

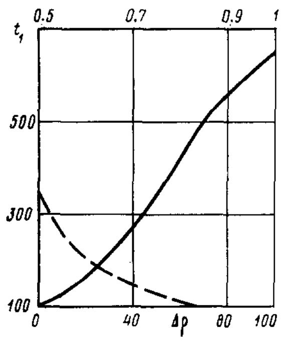

Fig.2

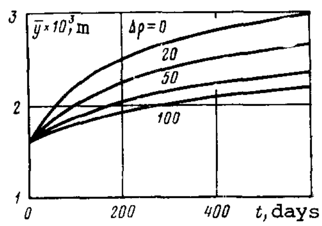

Fig.3

Values of the critical time t_1 (in days) are presented in Fig.2 as a function of the difference in the ages of the bar pieces $\Delta\rho$ (in days) for $\beta = 0.5$ (solid line), as is also the dependence of the critical time t_1 on the degree of reinforcement β (dashes). The function is $\rho(x) = 50$ days for $0 \leqslant x \leqslant 0.5$ and $\rho(x) = 0$ for $0.5 \leqslant x \leqslant 1$ for the dashed line. Results of computations showed that the critical time increases both as the difference in the ages increases and as the degree of reinforcement increases.

The dependence of the deflection value $\bar{y}(t)$ that is maximum in x is presented in Fig.3 in the form of a function of time t (in days) for $\beta = 0.5$. The curves correspond to different values of the difference in the ages $\Delta\rho$. The quantity $\Delta\rho$ varied between 0 for the upper curve and 100 days for the lower curve.

5. **Remark**. The method developed above for the investigation of the stability of viscoelastic reinforced bars is also applicable for certain other situations. In every specific case presented above, the equation for the deflections has the form (2.9), but the boundary conditions change as a function of the method of fixing the bar ends. Bar stability is examined in an infinite time interval in the sense of (3.1).

1°. There is a viscoelastic reinforced bar subjected to its own weight, and under moving support of the upper end and hinged support of the lower end. The boundary conditions for the deflection have the form

$$y'(t,\ 0) = 0,\quad y''(t,\ 0) = 0,\quad y''(t,\ l) = 0$$

The stability condition is given by formula (3.12) in which

$$\lambda_0 = 3.524\ l^{-3}$$

2°. We consider a bar with clamped lower end and moving support of the upper end. The boundary conditions have the form

$$y'(t,\ 0) = 0,\quad y''(t,\ 0) = 0,\quad y'(t,\ l) = 0$$

The bar is subjected to a weight with a constant longitudinal load g. The stability condition has the form (3.12) for $\lambda_0 = 18.99 l^{-3}$.

3°. Let the bar studied in Sect.3 be subjected to a longitudinal force P at the upper end. The mangnitude of the moment is $Py(t,\ x)$. The value of the critical force is given by the right side of (3.12) for $\lambda_0 = 4\pi^2 l^{-2}$.

The authors are grateful to V.V. Metlov for performing the computations.

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Translated by M.D.F.